

# MY CONTACT HOMOLOGY SHOPPING LIST

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ABSTRACT. We list the properties of contact homology, beyond purely formal, needed for the proofs of some of the recent applications of contact homology in dynamics to work. The list is put together for the AIM Transversality in Contact Homology Workshop.

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## 1. INTRODUCTION

In this write-up for the AIM Transversality in Contact Homology Workshop, we identify the properties of contact homology, beyond purely formal, needed for the proofs in [GGM, GH<sup>2</sup>M] to work. We also expect these properties to be useful for other applications of contact homology in dynamics; see, e.g., [Gü, HM].

The main result of [GH<sup>2</sup>M] is the “SDM theorem” asserting that under natural additional conditions the presence of a simple closed Reeb orbit of a particular type (the so-called SDM) implies the existence of infinitely many simple closed Reeb orbits. As an application, we (re)prove the existence of at least two closed Reeb orbits for any contact form supporting the standard contact structure on  $S^3$ ; see also [CGH, GGo, LL] for other proofs of this or for more general results. In [GGM], we prove a variant of the Conley conjecture for Reeb flows on the pre-quantization circle bundles over aspherical symplectic manifolds and then use this fact to establish the existence of infinitely many simple closed orbits for a low energy

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twisted geodesic flow on a surface of positive genus with non-vanishing magnetic field. We refer the reader to [GG14, Sect. 4 and 5] for a much more detailed discussion of these results.

In [GH<sup>2</sup>M], we work exclusively with the linearized contact homology, and hence one should be able to reprove the results of that paper relying instead on the equivariant symplectic homology and thus bypassing the transversality problems; cf. [BO12]. (Such as translation of the proof to a different language is however rather non-trivial.) On the other hand, in [GGM], it is essential to use mainly the cylindrical contact homology with its additional grading by the free homotopy classes of loops. (The SDM theorem from [GH<sup>2</sup>M] also enters the proof as an ingredient. It is interesting to note that, as of this writing, the SDM theorem does not have, due to the index restrictions, an analog relying on the cylindrical contact homology. Working with twisted geodesic flows, one can circumvent the use of cylindrical contact homology by utilizing a very particular filling.)

Throughout this write-up, we will focus on the cylindrical contact homology. There are two reasons for this. First of all, this is the type of contact homology where it is not clear how to get around the transversality problems. Secondly, much of what we say readily translates to the linearized contact homology.

It might be worth pointing out that for the (finite energy) holomorphic curves, i.e., when the transversality problem is left aside, the properties we list below are either well known or seem to present no serious difficulty to prove. However, it is not *a priori* clear that these properties would automatically carry over once the transversality problem is resolved and holomorphic curves are possibly replaced by some other objects. Finally, we note that our approach to the construction of the local contact homology is somewhat different, at least on the technical level, from that in [HM].

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## 2. CONTACT HOMOLOGY

**2.1. Setting.** In what follows,  $(M^{2n-1}, \xi)$  is a closed contact manifold. Let  $\alpha$  be a non-degenerate contact form with  $\ker \alpha = \xi$ . We assume that the periodic orbits of  $\alpha$  meet the necessary index conditions for the linearized contact homology  $HC_*(M, \xi)$  and the action-filtered linearized contact homology  $HC_*^I(M, \alpha)$ , where  $I = (a, b)$ , to be defined. (We will always require  $a$  and  $b$  to be outside the action spectrum  $\mathcal{S}(\alpha)$  of  $\alpha$ .) For the sake of simplicity, let us also assume that  $c_1(\xi) = 0$  in  $H^2(M; \mathbb{Z})$ .

The (cylindrical) contact homology complex  $CC_*(M, \alpha)$  is generated by the good orbits  $x$  of  $\alpha$ , graded by  $|x| = \mu_{CZ}(x) + n - 3$  and filtered by the action. The differential  $\partial: CC_*(M, \alpha) \rightarrow CC_{*-1}(M, \alpha)$  depends on some auxiliary data and has the form

$$\partial x = \sum_y m(x, y) y,$$

where  $m(x, y)$  counts certain maps  $u: S^1 \times \mathbb{R} \rightarrow M$  or some other geometric objects with signs and weights. In what follows, for the sake of simplicity, we assume

that  $u$  is the  $M$ -component of a map  $S^1 \times \mathbb{R} \rightarrow M \times \mathbb{R}$  which must satisfy some (translation invariant) equation or more generally certain conditions depending on some auxiliary structures and choices, which we actually do not quite know at this moment and which we call (CR) here. In the original definition of the contact homology, (CR) is the Cauchy–Riemann equation in the symplectization and, in addition, the solutions are required to have finite Hofer energy; see [EGH] and also [Bo]. For the sake of brevity, let us refer, somewhat unconventionally, to  $u$  as a Floer trajectory. Note that (CR) should make sense even when  $\alpha$  is degenerate although one can probably circumvent this requirement.

The  $\omega$ -energy of  $u$  is by definition

$$E_\omega(u) = \int_u d\alpha.$$

Clearly, the  $\omega$ -energy is translation invariant, and one essential feature of (CR) should be that

$$E_\omega(u) = 0 \text{ iff } u \text{ is a trivial Floer trajectory,} \quad (2.1)$$

i.e., a closed Reeb orbit. (See, e.g., [BEHWZ, Lemma 5.4] for a proof of (2.1) in the holomorphic case.) Furthermore,

$$E(u) = \mathcal{A}_\alpha(x) - \mathcal{A}_\alpha(y),$$

where  $u$  is (partially) asymptotic to  $x$  and  $y$ . Here the action  $\mathcal{A}_\alpha$  is defined as

$$\mathcal{A}_\alpha(x) := \int_x \alpha.$$

Thus  $\partial$  is action decreasing. The complex  $CC_*(M, \alpha)$ , and hence the homology, is also graded by the free homotopy classes of loops in  $M$ . The filtered contact homology is defined, by continuity, even when  $\alpha$  is degenerate, provided that  $\alpha$  has perturbations meeting the index conditions; see Section 3.1.

**2.2. Spatial localization vs. energy localization.** In this section, we do not require  $\alpha$  to be non-degenerate or  $M$  to be compact. There may be a free homotopy class of loops, say  $\mathfrak{c}$ , or a collection of such classes fixed in the background.

In several instances we need to have lower bounds on the  $\omega$ -energy of Floer trajectories passing through a certain region to spatially localize low energy trajectories, i.e., to ensure that such trajectories are confined to the complement of the region. Here is one variant of such an assertion, which would probably cover all the instances where the localization has been used so far.

Let  $S$  be a compact subset of  $M$  and let  $A$  be the set of actions of closed Reeb orbits passing through  $S$ . (The set  $S$  is usually a hypersurface in  $M$ .) Fix a neighborhood  $N$  of  $S$ . Then we want to be able to say that there exists a constant  $\epsilon = \epsilon(S, N, \alpha) > 0$  such that

$$E_\omega(u) > \epsilon \quad (2.2)$$

for every  $u$  passing through  $S$ , provided that  $u$  is (partially) asymptotic to some  $x$  and  $y$  which are contained entirely in the complement of  $N$  and such that the action interval  $[\mathcal{A}_\alpha(y), \mathcal{A}_\alpha(x)]$  does not intersect  $A$ .

Moreover,  $\epsilon > 0$  can be taken so that (2.2) holds for every Floer trajectory for every contact form  $\alpha'$  which is  $C^\infty$ -close to  $\alpha$ . (We probably don't care if we exactly have  $\ker \alpha' = \ker \alpha$  or not.) It would also be useful to know that  $\epsilon$  depends only on  $\alpha|_N$ . This is, of course, a rather general assertion, and what is actually needed is its various particular cases as in, e.g., [GH<sup>2</sup>M]. When Floer trajectories

come from genuine holomorphic curves in the symplectization, (2.2) follows from, e.g., the variant of Gromov compactness proved in [Fi], similarly to an argument in [McL], combined with (2.1).

Here is a typical application. Assume that  $x$  and  $y$ , closed Reeb orbits of  $\alpha'$ , lie in a complement of  $N$  and that the action difference  $|\mathcal{A}_{\alpha'}(x) - \mathcal{A}_{\alpha'}(y)|$  is small. Then there are no Floer trajectories between  $x$  and  $y$  passing through  $S$  provided that there are no closed Reeb orbits of  $\alpha$  in the free homotopy class of  $x$  and  $y$  passing through  $S$ , i.e.,  $A = \emptyset$ . This argument can be used to show that  $\partial^2 = 0$  in the construction of local contact homology. Note, however, that the latter condition appears to be very difficult to check in general (except perhaps in very few cases such as completely integrable Reeb flows) unless there are non-trivial homotopy class or action restrictions. A variant of (2.2) is utilized in the proof of the SDM theorem in [GH<sup>2</sup>M].

**2.3. The shift map and the Lusternik–Schnirelmann theory.** The cylindrical contact homology has a degree-two downward (i.e., degree  $-2$ ) shift map  $D$  which is an analogue of the pairing with the generator of  $H^2(BS^1; \mathbb{Z})$  in the equivariant symplectic homology; see [BO09, Sect. 7.2]. It would be useful to know how  $D$  effects the spectral invariants of  $\alpha$ . More specifically, denote by  $c_w(\alpha)$  the spectral invariant of  $\alpha$  associated with  $w \in \mathrm{HC}_*(M, \xi)$ . (By definition, we set  $c_0 = -\infty$ .) Note that although we are not making any non-degeneracy assumptions,  $c_w(\alpha)$  is defined on the level of homology. It should be very much straightforward to show that, for any reasonable definition of  $D$ , we automatically have  $c_{D(w)}(\alpha) \leq c_w(\alpha)$ . However, what we want is the strict inequality, i.e., as in other versions of the Lusternik–Schnirelmann (LS) theory (cf. [GG09, Sect. 6]),

$$c_{D(w)}(\alpha) < c_w(\alpha) \quad (2.3)$$

when  $w \neq 0$  and all simple closed Reeb orbits of  $\alpha$  are isolated in the extended phase space. The latter condition is essential. In the non-degenerate case, (2.3) is a consequence of (2.1) once we know that  $D$  can be defined via counting solutions of the (CR) equation. Without non-degeneracy, one should in addition show that, roughly speaking,  $D = 0$  on the level of local contact homology. A variant of (2.3) for the ECH is used in, e.g., [CGH]; see also [Gü] for an application of the shift map in the context of equivariant symplectic or linearized contact homology. I am not aware of any situation where having (2.3) for cylindrical contact homology is crucial, but it is probably only a matter of time before we encounter one.

### 3. COBORDISMS AND CONTINUATION

**3.1. Generalities.** In this section, a contact manifold is a manifold equipped with a contact form rather than a contact structure. Let us assume that, as expected, a symplectic cobordism  $(V, \omega)$  from a closed contact manifold  $(M_1, \alpha_1)$  to another one  $(M_0, \alpha_0)$  induces a “continuation map” between filtered contact homology complexes when the forms are non-degenerate. The map on the level of the filtered homology is always defined when the end points of the action are outside  $\mathcal{S}(\alpha_1)$  and  $\mathcal{S}(\alpha_0)$ , regardless of whether the forms are non-degenerate or not. The continuation maps are again obtained by counting (finite energy) solutions of a certain equation in  $\hat{V}$  obtained from  $V$  by attaching the bottom and the top parts of the symplectizations of  $M_1$  and, respectively,  $M_0$ . This equation, which we call (CR)

again, depends on some extra structure, etc. Note that at this point we do not quite know what (CR) should be.

Doing dynamics, it is sometimes more convenient to think, by analogy with the Hamiltonian setting, in terms of monotone homotopies. Here is a formal definition. We say that a family of contact forms  $\alpha_s$ ,  $s \in [0, 1]$ , on  $M$  foliates a symplectic cobordism  $(V, \omega)$  from  $\alpha_1$  to  $\alpha_0$  if there exists a family of contact type embeddings  $j_s: M \rightarrow V$  smoothly foliating  $V$ , in the obvious sense, and such that  $j_s^* \omega = d\alpha_s$  on  $M$ . Note that then  $V$  is necessarily diffeomorphic to the cylinder  $M \times [0, 1]$  with  $\partial V$  comprising two parts:  $j_0(M)$  (the positive end) and  $j_1(M)$  (the negative end). Thus, with this convention, the family  $\alpha_s$  is “decreasing” and we have a map  $HC_*^I(\alpha_0) \rightarrow HC_*^I(\alpha_1)$  when the end points of  $I$  are outside  $\mathcal{S}(\alpha_0) \cup \mathcal{S}(\alpha_1)$ .

Then we have the following “stability result” for contact homology: the map  $HC_*^I(\alpha_0) \rightarrow HC_*^I(\alpha_1)$  is an isomorphism when the end points of  $I$  are outside  $\mathcal{S}(\alpha_s)$  for all  $s$ . Furthermore, let  $I_s$  be a family of intervals such that for every  $s$  the end points of  $I_s$  are outside  $\mathcal{S}(\alpha_s)$  for a family  $\alpha_s$  not necessarily foliating a cobordism. Then the contact homology spaces  $HC_*^{I_s}(\alpha_s)$  are isomorphic. In particular, the global contact homology depends only on the contact structure, the filtered contact homology is defined for degenerate forms, etc. This result is stated (without a proof and for the linearized contact homology) in [GH<sup>2</sup>M]. The proof differs from the Hamiltonian case, where a homotopy induces a map in both directions, but it is nonetheless almost entirely formal, up to one property of the continuation maps, and hence carries over to other types of contact homology theories.

This property is that when  $\alpha$  is non-degenerate and  $|\tau|$  is sufficiently small there should be a matching choice of the auxiliary data for the forms  $\alpha$  and  $(1 + \tau)\alpha$  resulting in an isomorphism between their contact homology complexes. On the level of the filtered contact homology this isomorphism should be equal to the one induced by the natural cobordism between these two forms.

When dealing with the cylindrical homology, one has to assume in addition that, say, for a dense set of  $s \in [0, 1]$  the forms  $\alpha_s$  foliating a cobordism have non-degenerate perturbations meeting the index conditions.

**3.2. Spatial localization for continuation maps.** In some cases, it is essential to know that low energy solutions of (CR) in  $\hat{V}$  cannot pass through a certain region in  $M$ . To be more specific, assume that a family  $\alpha_s$  foliates  $V$ . Furthermore, let, as in Section 2.2,  $S$  be a compact subset of  $M$  and  $N$  be a neighborhood of  $S$ . Denote by  $A$  the union of the action spectra of  $\alpha_0$  and  $\alpha_1$  for closed Reeb orbits passing through  $S$ . Then we need to know that there exists a constant  $\epsilon = \epsilon(S, N, \alpha_s) > 0$  such that for every continuation trajectory  $u$  passing through  $S$  and (partially) asymptotic to some  $x$  and  $y$  we have

$$\mathcal{A}_{\alpha_0}(x) - \mathcal{A}_{\alpha_1}(y) > \epsilon, \quad (3.1)$$

provided that  $x$  and  $y$  are contained entirely in the complement of  $N$  and that the action interval  $[\mathcal{A}_{\alpha_1}(y), \mathcal{A}_{\alpha_0}(x)]$  does not intersect  $A$ .

One point to keep in mind here is that the auxiliary structure on  $\hat{V}$  we use here, and hence (CR), are adapted to the family  $\alpha_s$ .

Above, we may again have a free homotopy class of loops in  $M$  or a collection of such classes fixed in the background. Moreover, we would need to know that  $\epsilon > 0$  can be taken so that (3.1) still holds when the family  $\alpha_s$  is replaced by another

foliating family of contact forms  $\alpha'_s$  which is  $C^\infty$ -close to  $\alpha_s$ , and, ideally, that  $\epsilon$  depends only on  $\alpha_s|_N$ .

Of course, what we really need are some particular cases of (3.1). In [GH<sup>2</sup>M], we used the case of (3.1) (for the linearized contact homology) with  $N = S^1 \times N_0$ , where  $N_0$  is a spherical shell in  $\mathbb{R}^{2n}$  and  $\alpha_s = \lambda + c(s) dt$  and  $S = S^1 \times S^{2n-1}$ . Here  $\lambda$  is a primitive of the standard symplectic structure on  $\mathbb{R}^{2n}$ ,  $c(s)$  is a decreasing family of constants, and  $t$  is a coordinate on  $S^1$ . Then (3.1) holds for holomorphic curves when the almost complex structure on  $V$  comes from the standard complex structure on  $N_0$ ; see [GH<sup>2</sup>M].

Another instance where a variant of (3.1) can be used is the invariance of the local contact homology. In this setting,  $N$  is a thickened boundary of an isolating tubular neighborhood of a closed Reeb orbit  $x$ . We would prefer to set  $\alpha_s = \alpha$  on  $N$ , but a constant family does not foliate a cobordism. Instead, one can take a small perturbation of the constant family:  $\alpha_s = c(s)\alpha$ , where  $c(s)$  is a  $C^2$ -small monotone decreasing function on  $[0, 1]$ . (The argument in [HM] is different.)

It is worth pointing out that (3.1) has probably never been proved in detail in the general form as stated above for holomorphic curves, but from the first glance it looks correct.

#### 4. LOCAL CONTACT HOMOLOGY

**4.1. Definitions and basic properties.** The local contact homology is associated to an isolated (in the extended phase space) closed Reeb orbit. To be more specific, let  $x$  be such an orbit of the Reeb flow of  $\alpha$ . We do not assume that  $x$  is non-degenerate or simple. Under a small non-degenerate perturbation of  $\alpha$  the orbit  $x$  splits into a finite collection of non-degenerate orbits  $x'_i$  contained in a small isolating neighborhood  $U$  of  $x$ , and one can form a contact homology complex using  $x'_i$  with the differential defined exactly as in the global case. We denote the resulting homology by  $\mathrm{HC}_*(x)$  or  $\mathrm{HC}_*(\alpha, x)$ . The local contact homology is invariant under deformations  $\alpha_s$  of  $\alpha$  as long as  $x$  is uniformly isolated, i.e., having a common isolating neighborhood for all forms  $\alpha_s$ . The local contact homology is discussed in detail in [HM] and then, in lesser detail, in [GH<sup>2</sup>M]. Our outline of the proof that this homology is defined and well-defined differs in a number of ways from the proofs in [HM].

When the orbit  $x$  is simple, there are no transversality problems in the definition of the local contact homology because all orbits in question belong to a simple free homotopy class. In this case, we have

$$\mathrm{HC}_{*+n-3}(x) = \mathrm{HF}_*(\varphi), \quad (4.1)$$

where on the right we have the local Floer homology of (the germ of) the Poincaré return map  $\varphi$  of  $x$ ; see, e.g., [GG09] for the definition. A variant of this identity is essentially contained already in [EKP, Sect. 6]; see also [HM] for another proof. Note that in this case one can work over any coefficient ring.

When  $x$  is iterated, (4.1) is no longer true. In this case, the right hand side should be replaced by the equivariant symplectic homology. To be more precise, assume  $x = z^k$  where  $z$  is simple. Then the Poincaré return map  $\varphi$  of  $x$  is the  $k$ th iteration of the Poincaré return map of  $z$  and the  $\mathbb{Z}_k$ -equivariant local Floer homology  $\mathrm{HF}_*^{\mathbb{Z}_k}(\varphi)$  of  $\varphi$  is defined. We should have

$$\mathrm{HC}_{*+n-3}(x) = \mathrm{HF}_*^{\mathbb{Z}_k}(\varphi), \quad (4.2)$$

where both groups are taken over  $\mathbb{Q}$ ; cf. [BO12]. Now the definition of the left hand side encounters the usual transversality problems, and the definition of  $\mathrm{HC}_*(x)$  and the proof of (4.2) should rely on a version of abstract perturbations.

Let us ignore for a moment the transversality issue. Thus we assume that there exists an almost complex structure on the symplectization of  $U$  meeting all the regularity assumptions. This is, roughly speaking, equivalent to assuming that there exists a one-periodic in time almost complex structure on a local cross section to  $x$ , which is regular for the  $k$ -periodic map  $\varphi$ . In this case, as is observed in [GH<sup>2</sup>M], one can replace the equivariant homology  $\mathrm{HF}_*^{\mathbb{Z}_k}(\varphi)$  by the  $\mathbb{Z}_k$ -invariant homology  $\mathrm{HF}_*(\varphi)^{\mathbb{Z}_k}$ , and the equality

$$\mathrm{HC}_{*+n-3}(x) = \mathrm{HF}_*(\varphi)^{\mathbb{Z}_k} \quad (4.3)$$

can then be proved along with (4.2) by reasonably conventional methods as in, e.g., [EKP]. (The argument is outlined in [GH<sup>2</sup>M].)

Overall, the situation here is rather similar to equating, as in [BO12], the linearized contact homology and the equivariant symplectic homology, and perhaps is even a bit simpler than that. Some parts of the proof are given in [GH<sup>2</sup>M].

Although the identification (4.2) is illuminating, it has not been used anywhere to the best of the author's knowledge. What has been used is a weaker result that  $\dim \mathrm{HC}_*(x^k)$  is a bounded function of  $k$  as long as  $x^k$  remains isolated; see [HM].

Finally, the local contact homology are the building blocks for the global and filtered contact homology. Namely, assume that the only point of the action spectrum  $\mathcal{A}(\alpha)$  in the interval  $I$  is  $c$  and that all orbits  $x$  with action equal to  $c$  are isolated. Then we should have

$$\mathrm{HC}_*^I(M, \alpha) = \bigoplus_x \mathrm{HC}_*(x). \quad (4.4)$$

As a consequence,  $\mathrm{HC}_m^I(M, \alpha) = 0$  for any interval  $I$  when all orbits  $x$  with action in  $I$  are isolated and have zero local contact homology in degree  $m$ . Here, as in Section 2, we can fix a free homotopy class of loops in  $M$  or a collection of such classes.

There are several ways to circumvent the transversality issues in the construction of the local contact homology. For instance, one can simply declare (4.2) to be the definition of  $\mathrm{HC}_*(x)$ . However, the following approach proposed by Michael Hutchings is from our perspective more aesthetically pleasing. Namely, when  $x = z^k$  where  $z$  is simple, there is a natural  $\mathbb{Z}_k$ -action by contactomorphisms on the  $k$ -fold covering of a neighborhood of  $z$ . Combined with the continuation maps, this action gives rise to a  $\mathbb{Z}_k$ -action on  $\mathrm{HC}_*(\tilde{z})$ , where  $\tilde{z}$  is the  $k$ -fold covering of  $z$ , and one can just set  $\mathrm{HC}_*(x) = \mathrm{HC}_*(\tilde{z})^{\mathbb{Z}_k}$ . (The composition of a deck transformation and the continuation map is still  $k$ -periodic on the level of homology since continuation maps are canonical.) A similar construction can be used in the Hamiltonian setting to define  $\mathrm{HF}_*(\varphi)^{\mathbb{Z}_k}$  without assuming the existence of a regular, one-periodic in time almost complex structure. Then (4.3) becomes a consequence of (4.1) and the definitions. Note however that with any of these “indirect” definitions of the local contact homology, (4.4) and other similar facts would become much less obvious.

**4.2. Technicalities.** Regardless of the transversality issues, there are two other problems one has to deal with when defining the local contact homology, and this is where (2.2) and (3.1) can also be useful. (In [HM] a different approach, at least



on the technical level, is used to show that the local Floer homology is defined and, moreover, well-defined. That approach is likely to result in somewhat different requirements on the solutions of the (CR) equation than (2.2) and (3.1).)

The first problem is that to have  $\partial^2 = 0$  we need to show that the Floer trajectories for a small non-degenerate perturbation  $\alpha'$  of  $\alpha$  asymptotic to some of the orbits  $x$  splits into cannot leave  $U$ . The  $\omega$ -energy of such a trajectory  $u$  is necessarily small since  $\alpha'$  is close to  $\alpha$ . Then it follows immediately from (2.2) that  $u$  is confined to  $U$ .

The second problem is to show that the resulting homology is independent of the perturbation  $\alpha'$  and whatever auxiliary data is used. Here again the main point is to localize the trajectories to a small neighborhood of  $x$  by, say, using a variant of (3.1). Once this is done, the proof is identical to the stability argument for the global or filtered contact homology pointed out in Section 3.1.

In the context of the holomorphic curves, the proofs of both facts do not require any new machinery and are in part given above.

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